

MATHEMATISCH CENTRUM

2e BOERHAAVESTRAAT 49

AMSTERDAM

On a theorem of Shidlovski

by

K. Mahler

December 1959

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Part I: Algebraic preliminaries.

1. Notation.

Let k , the constant field, be any field of characteristic 0, z an indeterminate, and c an arbitrary constant (i.e. an element of k). As usual, $k[z]$, and $k(z)$, denote the ring of all polynomials, and the field of all rational functions in z with coefficients in k , respectively. Further $k_c(z)$ denotes the subset of $k(z)$ consisting of those rational functions $r = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials that are relatively prime and where, moreover, $q(c) \neq 0$. It is clear that $k_c(z)$ is a ring. The transformation $z \rightarrow c$ maps the elements of $k_c(z)$ onto the elements of k , but loses its meaning for the other elements of $k(z)$.

The degree of $r = \frac{p(z)}{q(z)}$ will be defined as the sum of the degrees of the polynomials $p(z)$ and $q(z)$ which are again assumed to be relatively prime. For polynomials $p(z) = \frac{p(z)}{1}$ this agrees with the usual definition. However, the rational function 0 which vanishes identically is not given any degree.

Next we denote by K_c the set of all formal power series

$$\alpha = a_0 + a_1(z-c) + a_2(z-c)^2 + \dots$$

with coefficients in k . On defining addition and multiplication in K_c as is usual for power series, K_c becomes a ring. This ring contains all polynomials in $k[z]$ and also all rational functions in $k_c(z)$; for the elements of $k_c(z)$ may be developed into such power series by simple division, or by using the formal analogue of Taylor's series. For we may define formal differentiation in K_c by

$$\frac{d\alpha}{dz} = \alpha' = 1 \cdot a_1 + 2 \cdot a_2(z-c) + 3 \cdot a_3(z-c)^2 + \dots,$$

and then the usual rules of differentiation remain valid. Naturally differentiation is also possible for the elements in $k[z]$ and $k(z)$ and agrees for the elements in $k[z]$ and $k_c(z)$ with that in K_c .

The power series α is said to be of order n if

$$a_0 = a_1 = \dots = a_{n-1} = 0, \text{ but } a_n \neq 0,$$

and we then write

$$\text{ord } \alpha = n.$$

The power series 0 with all coefficients equal to zero is not given any order.

2. Systems of formal linear differential equations.

Let

$$(1) \quad y'_k = \sum_{\kappa=1}^m a_{k\kappa}(z)y_{\kappa} \quad (k=1,2,\dots,m)$$

be a system of formal homogeneous linear differential equations where the coefficients

$$a_{k\kappa}(z) = a_{k\kappa 0} + a_{k\kappa 1}(z-c) + a_{k\kappa 2}(z-c)^2 + \dots$$

(k, κ = 1, 2, ..., m)

are given series in K_c . It is not difficult to find all solutions of this system of equations in terms of series in K_c . For shortness of notation we introduce the column vector

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \quad \text{with the derivative } y' = \begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_m \end{pmatrix}$$

and the square matrix

$$q(z) = \begin{pmatrix} a_{11}(z) & \dots & a_{1m}(z) \\ \vdots & & \vdots \\ a_{m1}(z) & \dots & a_{mm}(z) \end{pmatrix};$$

thus (1) becomes the matrix equation

$$(1^*) \quad y' = q(z)y.$$

The problem is then to find all vectors y of the form

$$(2) \quad y = y^{(0)} + y^{(1)}(z-c) + y^{(2)}(z-c)^2 + \dots$$

where $y^{(0)}, y^{(1)}, y^{(2)}, \dots$ are constant vectors (i.e. their components lie in k) that satisfy (1*). The matrix $q(z)$ has the analogous development

$$(3) \quad q(z) = q^{(0)} + q^{(1)}(z-c) + q^{(2)}(z-c)^2 + \dots$$

where $q^{(0)}, q^{(1)}, q^{(2)}, \dots$ are the constant matrices

$$q^{(n)} = \begin{pmatrix} q_{11n} \dots q_{1mn} \\ \vdots \\ q_{m1n} \dots q_{mmn} \end{pmatrix}.$$

On substituting the series (2) and (3) in (1*), we get the matrix relation

$$\begin{aligned} y' &= 1 \cdot y^{(1)} + 2y^{(2)}(z-c) + 3y^{(3)}(z-c)^2 + \dots = \\ &= (q^{(0)} + q^{(1)}(z-c) + q^{(2)}(z-c)^2 + \dots)(y^{(0)} + y^{(1)}(z-c) + y^{(2)}(z-c)^2 + \dots) \end{aligned}$$

Here we compare on both sides the coefficients of equal powers of $z-c$ and so find the following system of relations:

$$\begin{aligned} (4): \quad & 1 \cdot y^{(1)} = q^{(0)} y^{(0)} \\ & 2 \cdot y^{(2)} = q^{(1)} y^{(0)} + q^{(0)} y^{(1)} \\ & 3 \cdot y^{(3)} = q^{(2)} y^{(0)} + q^{(1)} y^{(1)} + q^{(0)} y^{(2)} \\ & \dots \end{aligned}$$

These recursive formulae allow to determine successively and uniquely the constant vectors $y^{(1)}, y^{(2)}, y^{(3)}, \dots$ when $y^{(0)}$ is any given constant vector, and there are clearly no other solutions.

Choose, in particular, for $y^{(0)}$ that vector which has as its k -th component 1 and has all other components equal to zero. The resulting solution of (1*) will be denoted by

$$Y_k = \begin{pmatrix} Y_{1k} \\ Y_{2k} \\ \vdots \\ Y_{mk} \end{pmatrix},$$

and we also introduce the square matrix

$$Y = \begin{pmatrix} Y_{11} \dots Y_{1m} \\ \vdots \\ Y_{m1} \dots Y_{mm} \end{pmatrix} = (Y_1, \dots, Y_m)$$

of which the vectors Y_k form the different columns.

From the linearity of the recursive formulae (4) it is evident that if, explicitly, the constant vector $y^{(0)}$ has the components

$$y(0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

where $c_1, c_2, \dots, c_m$ are arbitrary elements of k , the corresponding solution y of (1^*) becomes

$$(5) \quad y = c_1 Y_1 + \dots + c_m Y_m.$$

This is therefore the most general solution of (1^*) if $c_1, \dots, c_m$ run independently over all elements of k .

The vectors $Y_1, \dots, Y_m$ are linearly independent over k . For in the matrix Y the series of all diagonal elements begin with the constant term 1, while the constant terms of the elements outside the diagonal all vanish. Therefore the determinant of Y is a power series of the form

$$\det Y = 1 + d_1(z-c) + d_2(z-c)^2 + \dots (d_1, d_2, \dots \in k)$$

and so does not vanish identically. But then all the components of $c_1 Y_1 + \dots + c_m Y_m$ can only vanish if $c_1 = \dots = c_m = 0$, as asserted.

We call $Y_1, \dots, Y_m$ a fundamental system of solutions of (1^*) and say that Y is a fundamental matrix solution. Evidently Y and its derivative Y' satisfy the matrix equation

$$(1^{**}) \quad Y' = q(z)Y.$$

The most general matrix Y satisfying this equation is of the form

$$Y^* = Y C$$

where

$$C = \begin{pmatrix} c_{11} & \dots & c_{1m} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mm} \end{pmatrix}$$

is a constant matrix. This follows because the columns of Y^* are solutions of (1^*) and hence are of the form (5). We also see that Y^* is then and only then non-singular if

$$\det C \neq 0.$$

4. A lemma on orders.

Lemma 1: Let $\varphi_1, \varphi_2, \dots, \varphi_m$ be finitely many elements of K_0 . There exists a positive integer N_0 as follows: If $P_1, \dots, P_m$ are elements of k such that

$$R = P_1 \varphi_1 + \dots + P_m \varphi_m$$

does not vanish identically, then

$$\text{order } R \leq N_0.$$

Proof: We assume the assertion is false and deduce a contradiction.

The assumption means that there exist infinitely many systems

$$(P_{k1}, \dots, P_{km}) \quad (k=1, 2, 3, \dots)$$

of elements of k and an infinite sequence of integers n_k satisfying

$$1 \leq n_1 < n_2 < n_3 < \dots$$

such that, for all suffixes k ,

$$R_k = \sum_{\kappa=1}^m P_{k\kappa} \varphi_{\kappa} \neq 0, \quad \text{but} \quad \text{ord } R_k = n_k.$$

We consider the R_k as linear forms in $\varphi_1, \dots, \varphi_m$. Assume that certain l , but not more than l , of the forms R_k are linearly independent over k ; let this be the forms

$$R_{k_1}, R_{k_2}, \dots, R_{k_l} \quad \text{where } 1 \leq k_1 < k_2 < \dots < k_l.$$

Since these forms are independent, every other form R_k can be written as a sum

$$R_k = \sum_{\kappa=1}^l c_{k\kappa} R_{k_{\kappa}}$$

where the coefficients $c_{k\kappa}$ are in k and do not all vanish. Let, say, $c_{k\lambda} \neq 0$ where λ is chosen as the smallest suffix of this kind. Then evidently

$$\text{ord } R_k = \text{ord } R_{k_{\lambda}} = n_{k_{\lambda}} = n_{k_1},$$

and so a contradiction arises if $k > n_{k_1}$.

Lemma 2: Let $\varphi_1, \dots, \varphi_m$ be arbitrary elements in K_0 , and let $n \geq 0$ be an arbitrary integer. There exists a positive integer N_n as follows: If $P_1, \dots, P_m$ are arbitrary polynomials in $k[z]$ of degrees not exceeding n such that

$$R = P_1 \varphi_1 + \dots + P_m \varphi_m$$

does not vanish identically, then
ord $R \leq N$.

Proof: The assertion follows immediately on applying Lemma 1 to the $m(n+1)$ elements of k ,

$$\varphi_{hk} = z^h \varphi_k \quad (h=0,1,\dots,n; k=1,r,\dots,m).$$

5. A lemma on quotients in K_c .

While K_c is only a ring, it can easily be extended to a field $\mathcal{K}_c$ by allowing terms in negative powers of $r-c$; $\mathcal{K}_c$ consists of all series of the form

$$a_f(z-c)^f + a_{f+1}(z-c)^{f+1} + a_{f+2}(z-c)^{f+2} + \dots$$

with coefficients in k ; here f can be any integer greater than or equal to or less than zero. The field $\mathcal{K}_c$ so defined contains both $k[z]$ and $k(z)$ as subsets.

For functions in K_c the following property holds.

Lemma 3: Let $\varphi_1, \dots, \varphi_s, \psi_1, \dots, \psi_t$ be elements of K_c . There exists a positive integer N as follows. If $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_t$ are elements of k such that

- (i) $\alpha_1 \varphi_1 + \dots + \alpha_s \varphi_s \neq 0$ and $\beta_1 \psi_1 + \dots + \beta_t \psi_t \neq 0$,
- (ii) $\omega = \frac{\alpha_1 \varphi_1 + \dots + \alpha_s \varphi_s}{\beta_1 \psi_1 + \dots + \beta_t \psi_t} \neq 0$ is a rational function, then the

degree of ω does not exceed N .

Proof: The proof is again indirect. We shall assume there are infinitely many sets of $s+t$ elements

$$(\alpha_{k1}, \dots, \alpha_{ks}, \beta_{k1}, \dots, \beta_{kt}) \quad (k=1,2,3,\dots)$$

of k , and an infinite sequence of integers n_k satisfying

$$1 \leq n_1 < n_2 < n_3 < \dots$$

such that, for all k ,

- (i) $\alpha_{k1} \varphi_1 + \dots + \alpha_{ks} \varphi_s \neq 0$, $\beta_{k1} \psi_1 + \dots + \beta_{kt} \psi_t \neq 0$
- (ii) $\omega_k = \frac{\alpha_{k1} \varphi_1 + \dots + \alpha_{ks} \varphi_s}{\beta_{k1} \psi_1 + \dots + \beta_{kt} \psi_t}$ is an element of $k(z)$,
- (iii) ω_k has the exact degree n_k .

Of the series $\psi_1, \dots, \psi_t$ let r , but not more, be linearly independent over $k(z)$; without loss of generality

$$1 \leq r \leq t.$$

The numbering may be chosen such that

$$\psi_1, \dots, \psi_r$$

are linearly independent, while any remaining ψ_τ are linearly dependent on these r series.

It follows then that there exist $r(t-r)$ elements

$$D_{\tau\rho} \quad \left(\begin{array}{l} \tau = r+1, \dots, t \\ \rho = 1, 2, \dots, r \end{array} \right)$$

of $k(z)$ such that

$$\psi_\tau = \sum_{\rho=1}^r D_{\tau\rho} \psi_\rho \quad (\tau = r+1, r+2, \dots, t).$$

Denote by D the least common denominator of all the rational functions $D_{\tau\rho}$, and by d the maximum of the degrees of the polynomials $D_{\tau\rho}$ for all τ and ρ .

We can now write

$$\sum_{\tau=1}^t \beta_{k\tau} \psi_\tau = \sum_{\rho=1}^r \left\{ B_{k\rho} + \sum_{\tau=r+1}^t \beta_{k\tau} D_{\tau\rho} \right\} \psi_\rho = \sum_{\rho=1}^r B_{k\rho} \psi_\rho$$

where, for shortness,

$$B_{k\rho} = \beta_{k\rho} + \sum_{\tau=r+1}^t \beta_{k\tau} D_{\tau\rho} \quad \left(\begin{array}{l} \rho = 1, 2, \dots, r \\ k = 1, 2, 3, \dots \end{array} \right).$$

Also these sums $B_{k\rho}$ are in $k(z)$, and it is obvious that the products

$$D B_{k\rho}$$

are again polynomials at most of degree d . The second half of the hypothesis (i) implies that, for each k , at least one rational function among

$$B_{k1}, \dots, B_{kr}$$

does not vanish.

By the definition of ω_k ,

$$\sum_{\sigma=1}^s \alpha_{k\sigma} \varphi_\sigma = \sum_{\rho=1}^r \omega_k B_{k\rho} \psi_\rho \quad (k=1, 2, 3, \dots).$$

Here the expression

$$\sum_{\sigma=1}^s \alpha_{k\sigma} \varphi_{\sigma}, = L_k \text{ say,}$$

is a linear form in $\varphi_1, \dots, \varphi_s$ with coefficients in k . Let 1 but not more of the linear forms $L_1, L_2, L_3, \dots$ be independent over k ; without loss of generality,

$$1 \leq l \leq s,$$

and it may be assumed that the special forms

$$L_{k_1}, L_{k_2}, \dots, L_{k_l}, \quad \text{where } 1 \leq k_1 < k_2 < \dots < k_l,$$

are linearly independent over k . Hence, for every k , there are elements

$$c_{k1}, \dots, c_{kl}$$

of k such that

$$L_k = \sum_{\lambda=1}^l c_{k\lambda} L_{k_{\lambda}}.$$

Since, by the construction,

$$L_k = \sum_{\rho=1}^r \omega_k B_{k\rho} \psi_{\rho},$$

it follows then that

$$\sum_{\rho=1}^r \omega_k B_{k\rho} \psi_{\rho} = \sum_{\lambda=1}^l c_{k\lambda} \sum_{\rho=1}^r \omega_{k_{\lambda}} B_{k_{\lambda}\rho} \psi_{\rho}.$$

In this identity, $\psi_1, \dots, \psi_r$ are, by hypothesis, linearly independent over $k(z)$; moreover, the factors

$$\omega_k B_{k\rho}, \quad \omega_{k_{\lambda}}, \quad B_{k_{\lambda}\rho}$$

are all in $k(z)$. The identity therefore implies the r separate relations

$$(4) \quad \omega_k B_{k\rho} = \sum_{\lambda=1}^l c_{k\lambda} \omega_{k_{\lambda}} B_{k_{\lambda}\rho} \quad \text{for } \rho=1, 2, \dots, r.$$

Since not all rational functions

$$B_{k1}, \dots, B_{kr}$$

vanish identically, we can choose a suffix

$$\rho = \rho(k) = R$$

such that

$$B_{kR} = B_{k\rho}(k) \neq 0.$$

Consider the equation (7) corresponding to this suffix $\rho=R$,

$$\omega_k B_{kR} = \sum_{\lambda=1}^1 c_{k\lambda} \omega_{k_\lambda} B_{k_\lambda R}.$$

On the left-hand side ω_k is a rational function of degree n_k , and B_{kR} is $\neq 0$ and such that the polynomial DB_{kR} is at most of degree d . On the right-hand side the $c_{k\lambda}$ are constants depending on k ; the ω_{k_λ} are l fixed rational functions; and the $B_{k_\lambda R}$ are again rational functions such that $DB_{k_\lambda R}$ are all polynomials at most of degree d . Denote by Ω the least common denominator of

$$\omega_{k_1}, \omega_{k_2}, \dots, \omega_{k_l}$$

and by δ the largest of the degrees of

$$\Omega \omega_{k_1}, \Omega \omega_{k_2}, \dots, \Omega \omega_{k_l}.$$

Then

$$D \Omega \sum_{\lambda=1}^1 c_{k\lambda} \omega_{k_\lambda} B_{k_\lambda R}$$

is a polynomial at most of degree $d+\delta$, and hence

$$\omega_k = \Omega^{-1} \cdot (DB_{kR})^{-1} \cdot D \Omega \sum_{\lambda=1}^1 c_{k\lambda} \omega_{k_\lambda} B_{k_\lambda R}$$

is clearly a rational function of bounded degree, contrary to

$$n_k \rightarrow \infty \text{ as } k \rightarrow \infty.$$

This concludes the proof.

6. First properties of systems of linear differential equations with rational coefficients.

From now on we deal with a fixed system of linear differential equations

$$(1) \quad y_k' = \sum_{\kappa=1}^m q_{k\kappa}(z) y_\kappa \quad (k=1, 2, \dots, m)$$

where the coefficients

$$q_{k\kappa}(z) \quad (k, \kappa=1, 2, \dots, m)$$

are assumed to be rational functions, i.e. elements of $k(z)$. We consider an arbitrary set of m polynomials

$$P_1, P_2, \dots, P_m,$$

i.e. elements of $k[z]$; let only the case be excluded when all these vanish identically. We then form the linear form

$$R = P_1 y_1 + \dots + P_m y_m$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

is any vector satisfying (1).

To fix the ideas, let T be the least common denominator of all the m^2 rational functions $q_{k\kappa}$, and let, for the present, c be any element of k , such that

$$T(c) \neq 0.$$

We can then assume that the vector y has components in K_c . The results in §2 give the full description of all solution of (1) in vectors of this kind.

Let the dash be again the sign of formal differentiation. We form successively the system of further linear forms

$$R_1=R, \quad R_2=TR_1', \quad R_3=TR_2', \dots$$

It is then clear from (1) that all these forms become

$$R_k = P_{k1} y_1 + \dots + P_{km} y_m$$

where the coefficients

$$P_{k\kappa} \quad (k=1,2,3,\dots; \kappa=1,2,\dots,m)$$

are again polynomials. In fact,

$$P_{1\kappa} = P_{\kappa} \quad (\kappa=1,2,\dots,m),$$

and the definition, together with the differential equations, clearly imply that

$$(8) \quad P_{k+1,\kappa} = T(P_{k\kappa}' + \sum_{\lambda=1}^m P_{k\lambda} q_{\lambda\kappa}) \quad \text{for all } k \text{ and } \kappa.$$

This construction of the forms R_k in $y_1, \dots, y_m$ naturally does not depend on the special vector y satisfying (1).

Our hypothesis implies that the linear forms

$$R_k = P_{k1}y_1 + \dots + P_{km}y_m \quad (k=1,2,3,\dots)$$

do not all vanish identically, in fact, already the first one is not the zero form. Assume that the l forms

$$R_1, R_2, \dots, R_l, \quad \text{but not the } l+1 \text{ forms}$$

$$R_1, R_2, \dots, R_{l+1}$$

are linearly independent over $k(z)$. There holds then a relation

$$(9) \quad R_{l+1} = r_1 R_1 + r_2 R_2 + \dots + r_l R_l$$

where $r_1, r_2, \dots, r_l$ are certain rational functions (elements of $k(z)$). On differentiating repeatedly and multiplying each time by T and using again (9) and the general formula $R_{k+1} = TR'_k$, it is evident that, for every suffix $k \geq l+1$,

$$R_k = r_{k1} R_1 + \dots + r_{kl} R_l$$

where $r_{k1}, \dots, r_{kl}$ are again rational functions. Our hypothesis implies therefore that

the rank over $k(z)$ of the infinitely many linear forms

$$R_1, R_2, R_3, \dots \quad \text{is exactly } l.$$

Conversely, if this rank is l , the first l forms $R_1, \dots, R_l$ are linearly independent, for otherwise the last remarks would lead to a lower rank.

From now on we assume that the polynomials $P_1, \dots, P_m$, not all identically zero, have been fixed, and that the rank of the linear forms $R_1, \dots, R_m$ is exactly l ; clearly

$$1 \leq l \leq m.$$

The forms $R_1, \dots, R_{l+1}$ are then related by an identity (9) with rational coefficients. Since

$$R_1 = R, \quad R_2 = TR'_1, \quad R_3 = TT'R'_1 + T^2R''_1 \dots$$

it follows then that $R = P_1y_1 + \dots + P_my_m$ satisfies a linear differential equation of the form

$$(10) \quad R^{(1)} + p_1 R^{(1-1)} + p_2 R^{(1-2)} + \dots + p_l R = 0$$

where $p_1, p_2, \dots, p_l$ are certain rational functions.

7. A lemma on linear differential equations.

Let $\varphi_1, \varphi_2, \dots, \varphi_m$ be m elements of K_c that are linearly independent over k . I assert that the Wronskian determinant

$$W(\varphi_1 \varphi_2 \dots \varphi_m) = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_m \\ \varphi_1' & \varphi_2' & \dots & \varphi_m' \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(m-1)} & \varphi_2^{(m-1)} & \dots & \varphi_m^{(m-1)} \end{vmatrix}$$

is not identically zero.

Since $W(\varphi_1) = \varphi_1$, the assertion is certainly true for $m=1$; assume it has already been proved for the Wronskian of $m-1$ functions where $m \geq 2$; the following proof establishes it then also for m functions.

The hypothesis implies that none of the φ 's vanishes identically; so, without loss of generality, let φ_m be that function which has lowest order. It follows that all quotients

$\frac{\varphi_\mu}{\varphi_m}$ ($\mu=1, 2, \dots, m-1, m$) are again in K_c . Put

$$\psi_\mu = \frac{d}{dz} \left(\frac{\varphi_\mu}{\varphi_m} \right) \quad (\mu=1, 2, \dots, m).$$

Then the ψ_μ are likewise in K_c , and

$$\psi_m = 0.$$

The remaining functions $\psi_1, \dots, \psi_{m-1}$ are linearly independent over k because an identity

$$c_1 \psi_1 + \dots + c_{m-1} \psi_{m-1} \equiv 0$$

with coefficients not all zero in k would, on integrating, imply that

$$\frac{d}{dz} \left(c_1 \frac{\varphi_1}{\varphi_m} + \dots + c_{m-1} \frac{\varphi_{m-1}}{\varphi_m} \right) \equiv 0, \quad c_1 \frac{\varphi_1}{\varphi_m} + \dots + c_{m-1} \frac{\varphi_{m-1}}{\varphi_m} \equiv -c_m$$

where $c_m \in k$, leading to a contradiction.

We have then, by the induction hypothesis,

$$W(\psi_1 \dots \psi_{m-1}) = \begin{vmatrix} \psi_1 & \dots & \psi_{m-1} \\ \psi_1' & \dots & \psi_{m-1}' \\ \vdots & & \vdots \\ \psi_1^{(m-2)} & \dots & \psi_{m-1}^{(m-2)} \end{vmatrix} \neq 0.$$

On the other hand, on putting

$$\chi_k = \frac{\varphi_k}{\varphi_m} \text{ so that } \psi_k = \chi_k', \chi_m = 1$$

we have

$$\varphi_k = \varphi_m \chi_k, \varphi_k' = \varphi_m' \chi_k + \varphi_m \chi_k', \varphi_k'' = \varphi_m'' \chi_k + 2\varphi_m' \chi_k' + \varphi_m \chi_k'', \dots$$

Therefore

$$W(\varphi_1 \dots \varphi_m) = \begin{vmatrix} \varphi_m \chi_1 & \dots & \varphi_m \chi_m \\ \varphi_m' \chi_1 + \varphi_m \chi_1' & \dots & \varphi_m' \chi_m + \varphi_m \chi_m' \\ \varphi_m'' \chi_1 + 2\varphi_m' \chi_1' + \varphi_m \chi_1'' & \dots & \varphi_m'' \chi_m + 2\varphi_m' \chi_m' + \varphi_m \chi_m'' \\ \vdots & & \vdots \\ \varphi_m^{(m-1)} \chi_1 + \binom{m-1}{1} \varphi_m' \chi_1' + \dots + \varphi_m \chi_1^{(m-1)} & \dots & \varphi_m^{(m-1)} \chi_m + \binom{m-1}{1} \varphi_m' \chi_m' + \dots + \varphi_m \chi_m^{(m-1)} \end{vmatrix}$$

or, simpler,

$$W(\varphi_1 \dots \varphi_m) = \varphi_m^m W(\chi_1 \dots \chi_m)$$

and here clearly

$$W(\chi_1 \dots \chi_m) = (-1)^m W(\psi_1 \dots \psi_{m-1}).$$

The assertion follows then at once.

We can now prove

Lemma 4: Let $m > 1$, and let $\varphi_1, \varphi_2, \dots, \varphi_m$ be solutions in K_c of the non-trivial differential equation

$$p_0 y^{(1)} + p_1 y^{(1-1)} + \dots + p_l y = 0$$

where $p_0 \neq 0$, $p_1, \dots, p_l$ are in K_c . Then $\varphi_1, \varphi_2, \dots, \varphi_m$ are linearly dependent over k .

Proof: If $\varphi_1, \dots, \varphi_m$ are linearly independent over k , so are in particular $\varphi_1, \dots, \varphi_{l+1}$, and hence

$$W(\varphi_1, \dots, \varphi_{l+1}) \neq 0.$$

On the other hand, $\varphi_1, \dots, \varphi_{l+1}$ satisfy the equations

$$p_0 \varphi_\lambda^{(1)} + p_1 \varphi_\lambda^{(1-1)} + \dots + p_l \varphi_\lambda = 0 \quad (\lambda = 1, 2, \dots, l+1),$$

and the determinant of these is $W(\varphi_1, \dots, \varphi_{l+1})$. Thus $p_0, p_1, \dots, p_l$ would all vanish identically, contrary to hypothesis.

3. Special solutions of the differential equation for R.

We continue with the discussion in §6 of R. We had put

$$R = P_1 y_1 + \dots + P_m y_m$$

where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ was any solution K_c of (1), and $P_1, \dots, P_m$ were polynomials such that exactly $R_1, \dots, R_l$ but not $R_1, \dots, R_{l+1}$ were linearly independent over $k(z)$. We found then that R satisfied a differential equation (10) where the coefficients are rational functions.

This differential equation is satisfied however the vector y was chosen.

We found, that the general solution of (1) could be expressed in terms of the fundamental matrix

$$Y = \begin{pmatrix} Y_{11} & \dots & Y_{1m} \\ \vdots & & \vdots \\ Y_{m1} & \dots & Y_{mm} \end{pmatrix}$$

in which all column vectors are solutions of (1), hence also all linear combinations of such columns with constant coefficients.

Put,

$$R_{1k} = P_1 Y_{1k} + \dots + P_m Y_{mk} \quad (k=1, 2, \dots, m)$$

and similarly

$$R_{hk} = P_{h1} Y_{1k} + \dots + P_{hm} Y_{mk} \quad (h, k=1, 2, \dots, m).$$

We know then that all m functions

$$R_{11}, R_{12}, \dots, R_{1m}$$

satisfy the equation

$$R^{(1)} + p_1 R^{(1-1)} + \dots + p_l R = 0.$$

If $l < m$, by lemma 4, it follows that constants $c_{11}, \dots, c_{m1}$ not all zero exist such that

$$c_{11} R_{11} + c_{21} R_{12} + \dots + c_{m1} R_{1m} \equiv 0.$$

Now

$$\sum_{\kappa=1}^m c_{\kappa 1} R_{1\kappa} = \sum_{\kappa=1}^m c_{\kappa 1} \sum_{k=1}^m P_k Y_{k\kappa} = \sum_{k=1}^m P_k Y_{k1}^* \equiv 0$$

where

$$Y_{k1}^* = \sum_{\kappa=1}^m c_{\kappa 1} Y_{k\kappa} \quad (k=1,2,\dots,m).$$

It is clear that at least one component of the vector

$$\begin{pmatrix} Y_{11}^* \\ \vdots \\ Y_{m1}^* \end{pmatrix}$$

does not vanish identically.

We can now show

Lemma 5: There exists a constant non-singular matrix

$$C = \begin{pmatrix} c_{11} & \dots & c_{1m} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mm} \end{pmatrix}$$

as follows: Put

$$Y^* = Y C = \begin{pmatrix} Y_{11}^* & \dots & Y_{1m}^* \\ \vdots & & \vdots \\ Y_{m1}^* & \dots & Y_{mm}^* \end{pmatrix}$$

so that also Y^* is a fundamental matrix solution of (1).

Then, for $1 < m$, the first $m-1$ expressions

$$R_{1k}^* = F_1 Y_{1k}^* + \dots + F_m Y_{mk}^* \quad (k=1,2,\dots,m)$$

vanish identically.

Proof: We found already that if $1 < m$, we could find the first column of C such that the corresponding R_{11}^* was zero. Assume now that $m-1 \geq 2$, and that, say, $c_{11} \neq 0$; this does not restrict the problem as we can, if necessary, change the order of the n functions $y_1, \dots, y_m$. The $m-1 > 1$ functions

$$R_{12}, R_{13}, \dots, R_{1m}$$

still satisfy (10) and so constants $c_{12}, c_{22}, \dots, c_{m2}$ with $c_{12} = 0$ exist such that

$$c_{12} R_{11}^* + \dots + c_{m2} R_{1m}^* \equiv 0,$$

that is $R_{12}^* \equiv 0$. Naturally the first two columns of C are linearly independent over k .

In this way we can continue exactly $m-1$ times and obtain the asserted result.

9. The rational functions D_{ij} .

We continue with the general discussion of

$$R = P_1 y_1 + \dots + P_m y_m$$

and the derived expressions

$$R_k = P_{k1} y_1 + \dots + P_{km} y_m \quad (k=1, 2, \dots, m)$$

where $P_1, \dots, P_m$ were polynomials not all zero. We assume again that $R_1, \dots, R_l$, but not $R_1, \dots, R_l, R_{l+1}$ are linearly independent over $k(z)$. This means that the square matrix

$$P = \begin{pmatrix} P_{11} & \dots & P_{1m} \\ \vdots & & \vdots \\ P_{m1} & \dots & P_{mm} \end{pmatrix}$$

has the rank l , and that in fact already the submatrix

$$\begin{pmatrix} P_{11} & \dots & P_{1m} \\ \vdots & & \vdots \\ P_{l1} & & P_{lm} \end{pmatrix}$$

has this rank. This submatrix contains then a minor

$$\Delta_0 = \begin{vmatrix} P_{1\lambda_1} & P_{1\lambda_2} & \dots & P_{1\lambda_l} \\ \vdots & \vdots & & \vdots \\ P_{l\lambda_1} & P_{l\lambda_2} & \dots & P_{l\lambda_l} \end{vmatrix} \quad (\lambda_1 < \lambda_2 < \dots < \lambda_l)$$

that does not vanish. Denote by

$$\kappa_1, \kappa_2, \dots, \kappa_{m-l} \quad (\kappa_1 < \kappa_2 < \dots < \kappa_{m-l})$$

all suffixes $1, 2, \dots, m$ ^{that} are distinct from $\lambda_1, \lambda_2, \dots, \lambda_l$ so that the λ 's and κ 's together are all integers $1, 2, \dots, m$.

In the matrix

$$\begin{pmatrix} P_{1\lambda_1} & \dots & P_{1\lambda_l} & P_{1\kappa_1} & \dots & P_{1\kappa_{m-l}} \\ \vdots & & \vdots & \vdots & & \vdots \\ P_{l\lambda_1} & \dots & P_{l\lambda_l} & P_{l\kappa_1} & \dots & P_{l\kappa_{m-l}} \end{pmatrix}$$

the first l columns are linearly independent over $k(z)$, and hence the $m-l$ last columns are linear combinations of the l first columns with coefficients in $k(z)$. In explicit form,

$$(11^*) \quad P_{k\kappa_j} = \sum_{i=1}^1 P_{k\lambda_i} D_{ij} \quad \begin{matrix} k=1,2,\dots,l \\ j=1,2,\dots,m-1 \end{matrix}$$

where the D_{ij} are certain rational functions in $k(z)$. Our next aim is to study these functions in more detail. Naturally such functions occur only if $l < m$.

We have for $k=1,2,\dots,l$,

$$\begin{aligned} R_k &= \sum_{\kappa=1}^m P_{k\kappa} y_{\kappa} = \sum_{i=1}^1 P_{k\lambda_i} y_{\lambda_i} + \sum_{j=1}^{m-1} P_{k\kappa_j} y_{\kappa_j} \\ &= \sum_{i=1}^1 P_{k\lambda_i} y_{\lambda_i} + \sum_{j=1}^{m-1} \sum_{i=1}^1 P_{k\lambda_i} D_{ij} y_{\kappa_j} = \sum_{i=1}^1 P_{k\lambda_i} u_i \end{aligned}$$

where, for shortness,

$$u_i = y_{\lambda_i} + \sum_{j=1}^{m-1} D_{ij} y_{\kappa_j} \quad (i=1,2,\dots,l).$$

When $l=m$, the u 's are simply all the y 's: $u_i=y_i$).

We repeat that

$$R_k = \sum_{i=1}^1 P_{k\lambda_i} u_i \quad (k=1,2,\dots,l).$$

We apply now lemma 5. Let the constant matrix C and the fundamental matrix solution $Y^*=YC$ be chosen according to this lemma. Then put

$$u_{ih}^* = Y_{\lambda_i h}^* + \sum_{j=1}^{m-1} D_{ij} Y_{\kappa_j h}^* \quad \left(\begin{matrix} i=1,2,\dots,l \\ h=1,2,\dots,m \end{matrix} \right)$$

so that

$$R_{kh}^* = \sum_{i=1}^m P_{ki} Y_{ih}^* \quad (h,k=1,2,\dots,m)$$

takes the form

$$R_{kh}^* = \sum_{i=1}^1 P_{h\lambda_i} u_{ih}^* \quad \left(\begin{matrix} k=1,2,\dots,l \\ h=1,2,\dots,m \end{matrix} \right)$$

The lemma tells us now that

$$R_{kh}^* \equiv 0 \quad \text{if } k=1,2,\dots,l; h=1,2,\dots,m-1.$$

For each fixed value of h , this is a system of homogeneous linear equations for

$$u_{1h}^*, u_{2h}^*, \dots, u_{lh}^*$$

with the determinant

$$\Delta_0 \neq 0.$$

It follows therefore that

$$(11) \quad u_{ih}^* = Y_{\lambda_i h}^* + \sum_{j=1}^{m-1} D_{ij} Y_{\kappa_j h}^* \equiv 0 \quad \left(\begin{array}{l} i=1,2,\dots,l \\ h=1,2,\dots,m-1 \end{array} \right)$$

This is a system of $l(m-1)$ equations for the $l(m-1)$ functions D_{ij} . We show that these equations determine the D_{ij} uniquely.

To prove this it suffices to prove that the determinant

$$\Lambda = \begin{vmatrix} Y_{\kappa_1 1}^* & \dots & Y_{\kappa_1 m-1}^* \\ \vdots & & \vdots \\ Y_{\kappa_{m-1} 1}^* & \dots & Y_{\kappa_{m-1} m-1}^* \end{vmatrix}$$

does not vanish identically.

We know that the determinant

$$\Sigma = \begin{vmatrix} Y_{\kappa_1 1}^* & \dots & Y_{\kappa_1 m-1}^* & Y_{\kappa_1 m-1+1}^* & \dots & Y_{\kappa_1 m}^* \\ \vdots & & \vdots & & & \vdots \\ Y_{\kappa_{m-1} 1}^* & \dots & Y_{\kappa_{m-1} m-1}^* & Y_{\kappa_{m-1} m-1+1}^* & \dots & Y_{\kappa_{m-1} m}^* \\ Y_{\lambda_1 1}^* & \dots & Y_{\lambda_1 m-1}^* & Y_{\lambda_1 m-1+1}^* & \dots & Y_{\lambda_1 m}^* \\ \vdots & & \vdots & \vdots & & \vdots \\ Y_{\lambda_l 1}^* & \dots & Y_{\lambda_l m-1}^* & Y_{\lambda_l m-1+1}^* & \dots & Y_{\lambda_l m}^* \end{vmatrix},$$

which differs from $|Y^*|$ by at most the sign, cannot vanish. We can change Σ as follows into a new determinant.

For each suffix $i=1,2,\dots,l$ multiply the first $m-1$ rows successively by the factors

$$D_{i1}, D_{i2}, \dots, D_{im-1}$$

and add to the row where the first suffix is λ_i . The identities (11) show that this operation changes Σ into a new determinant Σ^* where all elements in the rectangle formed by the last l rows and first $m-1$ columns vanish. If now also $\Lambda=0$, all minors of order $m-1$ of the first $m-1$ columns of Σ^* would be zero, and hence $\Sigma^*=0$, thus $\Sigma=0$, which is false.

This proves that

$$\Lambda \neq 0$$

so that the functions D_{ij} are fixed uniquely by the equations (11). On the other hand, by (11*), they depend only on the polynomials P_{ki} . As we can now show, the dependence on the latter is rather of

a simple kind.

The equations (11) can be solved for the D_{ij} and give the result that

$$D_{ij} = - \frac{\Lambda_{ij}}{\Lambda} \quad \left(\begin{array}{l} i=1,2,\dots,l \\ j=1,2,\dots,m-1 \end{array} \right)$$

where Λ_{ij} is the determinant obtained from Λ on replacing in it the row

$$Y_{\kappa_1 1}^* \cdots Y_{\kappa_1 m-1}^*$$

by the new row

$$Y_{\lambda_j 1}^* \cdots Y_{\lambda_j m-1}^* .$$

Now, by construction,

$$(12) \quad Y^* = YC ,$$

and here the constant matrix C depends on the polynomials P_{ki} . Denote by

$$\varphi_1, \varphi_2, \dots, \varphi_M$$

the set of all minors of order $m-1$ that can be found from Y .

It follows then that every minor of order $m-1$ of Y^* is of the form

$$\gamma_1 \varphi_1 + \gamma_2 \varphi_2 + \dots + \gamma_M \varphi_M$$

where the γ 's are elements of k ; they may in fact be written as sums of products of elements of C .

This representation holds in particular for the minors $\Lambda \neq 0$ and Λ_{ij} . Thus we obtain a representation

$$(13) \quad D_{ij} = \frac{\alpha_1^{(ij)} \varphi_1 + \alpha_2^{(ij)} \varphi_2 + \dots + \alpha_M^{(ij)} \varphi_M}{\beta_1^{(ij)} \varphi_1 + \beta_2^{(ij)} \varphi_2 + \dots + \beta_M^{(ij)} \varphi_M} \quad \left(\begin{array}{l} i=1,2,\dots,l \\ j=1,2,\dots,m-1 \end{array} \right)$$

where

$$\sum_{\mu=1}^M \beta_{\mu}^{(ij)} \varphi_{\mu} \neq 0$$

and the α 's and β 's are suitable elements in k .

In these formulae (13) the rank l enters decisively. The formulae become void when $l=m$ as there are then no D_{ij} ; but they are of fundamental importance when $1 \leq l < m$. Depending on l , different minors φ_j enter the formula; but these minors do not otherwise depend on $P_1, \dots, P_m$.

We are interested only in those D_{ij} that do not vanish iden-

tically. For these lemma 3 leads at once to

Lemma 6: Independent of the polynomials $P_1, \dots, P_m$ and the rank l (assumed $< m$), the degrees of all polynomials D_{ij} that do not vanish identically are bounded.

10. A sufficient condition for the rank of R to be m .
The proof of lemma 6 made use of the ring K_c where c was supposed to be chosen such that $T(c) \neq 0$. However, the polynomials D_{ij} , as defined originally, do not depend on c . So, having proved lemma 6, we can now again drop this restriction on c : From now on, c may or may not satisfy $T(c) = 0$.

We specialize, however, the vector solution y of (1) from now on. Let namely c be any fixed element of k , and let

$$y_1 = f_1, \dots, y_m = f_m$$

be a special solution of (1) with the following two properties:

- (i) $f_1, \dots, f_m$ are series in K_c .
- (ii) $f_1, \dots, f_m$ are linearly independent over $k(z)$.

The second condition implies, in particular, that none of $f_1, \dots, f_m$ vanishes identically.

Let again $P_1, \dots, P_m$ be m polynomials not all identically zero. We now put

$$R = P_1 f_1 + \dots + P_m f_m,$$

thus replace the y 's by the f 's. Then in the same way as before,

$$R_k = P_{k1} f_1 + \dots + P_{km} f_m \quad (k=1, 2, \dots, m).$$

We assume again that the matrix $(P_{k\alpha})$ has the exact rank l where

$$1 \leq l \leq m;$$

This means thus that the linear forms $R_1, \dots, R_l$ are linearly independent over $k(z)$, but the linear forms $R_1, \dots, R_{l+1}$ are not.

We further put

$$U = \max(\deg P_1, \dots, \deg P_m),$$

$$V = \text{ord } R.$$

Our aim is to prove that l must be m if V is sufficiently large compared with U .

As before $T(x)$ denotes the least common denominator of all rational functions q_{kx} . Let similarly T_1 be for $1 \leq m$ the least common denominator of all rational functions D_{1j} , but for $l=m$ be equal to 1.

$$\text{Put } L_i = T_1 u_i = T_1 \left(f_{\lambda_i} + \sum_{j=1}^{m-1} D_{1j} f_{\kappa_j} \right) \quad (i=1, 2, \dots, l).$$

Thus L_i is a linear form in $f_1, \dots, f_m$ with coefficients that are polynomials not all identically zero. The hypothesis (ii) implies therefore that no L_i vanishes identically. The expression

$$p = \min(\text{ord } L_1, \dots, \text{ord } L_l)$$

has then a good meaning; in the special case $l=m$ we have instead,

$$p = \min(\text{ord } f_1, \dots, \text{ord } f_m).$$

In either case, p does not exceed a constant independent of the choice of $P_1, \dots, P_m$. This follows for $1 \leq m$ from lemma 2 because the polynomials

$$T_1, T_1 D_{1j}$$

are all of bounded degree; for $l=m$ the assertion is obvious.

As p cannot then be arbitrarily large, let r_0 the largest value it can attain.

By our construction, the determinant

$$\Delta_0 = \begin{vmatrix} P_1 \lambda_1 & \dots & P_1 \lambda_l \\ \vdots & & \vdots \\ P_l \lambda_1 & \dots & P_l \lambda_l \end{vmatrix}$$

does not vanish identically. Further

$$R_k = \sum_{i=1}^l P_{k\lambda_i} u_i \quad (k=1, 2, \dots, l).$$

Denote by Δ_{ki} the cofactor of $P_{k\lambda_i}$ in Δ_0 . Then, on solving for u_i

$$\Delta_0 u_i = \sum_{k=1}^l \Delta_{ki} R_k \quad (i=1, 2, \dots, l)$$

and hence

$$\Delta_0 L_i = \sum_{k=1}^l T_1 \Delta_{ki} R_k \quad (i=1, 2, \dots, l).$$

In this system of equations choose the suffix i such that

$$\text{ord } L_i = p \leq r_0.$$

By hypothesis,

$$\text{ord } R = \text{ord } R_1 = V.$$

From $R_k = T R_{k-1}'$, where T is a polynomial, it follows that

$$\text{ord } R_k \geq V - k + 1 \quad (k=1, 2, \dots, l).$$

Further all factors $T_1 \Delta_{ki}$ are polynomials. The equation for $\Delta_0 L_i$ implies therefore that

$$\text{ord}(\Delta_0 L_i) \geq V - l + 1$$

and hence that

$$\text{ord } \Delta_0 \geq V - l + 1 - \text{ord } L_i \geq V - (r_0 + l - 1).$$

This inequality means that Δ_0 has the form

$$\Delta_0 = (z - c)^{V - (r_0 + l - 1)} \Delta_1$$

where Δ_1 is a second polynomial not identically zero. It also follows that

$$\deg \Delta_0 \geq V - (r_0 + l - 1)$$

We next establish also an upper bound for $\deg \Delta_0$. Let q be the largest of the degrees of all polynomials

$$T, Tq_{k\kappa}.$$

Since, by assumption,

$$\max(\deg P_1, \dots, \deg P_m) = U,$$

the recursive formulae for the $P_{k\kappa}$ lead at once to the inequalities

$$\deg P_{k\kappa} \leq U + (k-1)q.$$

Therefore, from its definition as a determinant,

$$\deg \Delta_0 \leq \sum_{k=1}^l (U + (k-1)q) = l U + \frac{(l-1)l}{2} q.$$

We finally combine the two bounds for $\deg \Delta_0$ and find that

$$V - (r_0 + l - 1) \leq l U + \frac{(l-1)l}{2} q$$

or

$$V - l U \leq \frac{(l-1)l}{2} q + (r_0 + l - 1)$$

or, by $l \leq m$,

$$V - l U \leq \Gamma$$

where Γ denotes the constant

$$\Gamma = \frac{m(m-1)}{2} q + (r_0 + m - 1).$$

The last inequality is satisfied if the form

$$R = P_1 f_1 + \dots + P_m f_m$$

has the exact rank 1. We therefore have the following final result:

Lemma 7: Let the system of linear differential equations

$$y'_k = \sum_{\kappa=1}^m a_{k\kappa} y_\kappa \quad (k=1, 2, \dots, m)$$

with rational coefficients $a_{k\kappa}$ have the special solution $y_1=f_1, \dots, y_m=f_m$ where $f_1, \dots, f_m$ are elements of K_0 that are linearly independent over $k(z)$. There exists a positive integer Γ depending only on $f_1, \dots, f_m$, as follows.

Let $P_1, \dots, P_m$ be any polynomials not all identically zero, and let

$$U = \max(\deg P_1, \dots, \deg P_m),$$

$$V = \text{ord } R \text{ where } R = R_1 = P_1 f_1 + \dots + P_m f_m.$$

Let further, recursively,

$$R_k = P_{k1} f_1 + \dots + P_{km} f_m = T R'_{k-1} \quad (k=2, 3, \dots)$$

where T is the least common denominator of the $a_{k\kappa}$.

Whenever

$$V - (m-1)U > \Gamma, \quad \text{the determinant}$$

$$\Delta = \begin{vmatrix} P_{11} & \dots & P_{1m} \\ \vdots & & \vdots \\ P_{m1} & & P_{mm} \end{vmatrix}$$

is not identically zero.

From now we shall restrict our discussion to the case when k is the complex number field, $c=0$, and all series are convergent for all z . It will in fact be necessary to restrict the functions $f(z)$ still further, and this is due entirely to reasons unconnected with the previous theory.

Let K be an arbitrary algebraic number field of finite degree h over the rational field. The power series

$$f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$$

is said to be an E-function if the following conditions hold:

- 1) All coefficients c_n are in K .
- 2) If $\varepsilon > 0$ is arbitrarily small and $\overline{c_n}$ denotes the maximum of the absolute values of c_n and all its conjugates with respect to K ,

$$\overline{c_n} = O(n^{n\varepsilon}).$$

Thus, in particular, $f(z)$ converges for all z .

- 3) For each $n=0,1,2,\dots$ there exists a smallest positive integer q_n such that all products

$$q_n^{c_0}, q_n^{c_1}, \dots, q_n^{c_n}$$

are algebraic integers. We assume that also

$$q_n = O(n^{n\varepsilon}).$$

By way of example,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is an E-function, and so is every polynomial with coefficients in K . If $f(z)$ is an E-function, so is $f'(z)$. One further shows without difficulty that sum, difference, and product of E-functions are E-functions, and that if $f(z)$ is an E-function and λ is any number in K , $f(\lambda z)$ is likewise an E-function.

We shall in ^{the} future always assume that the linear differential equations

$$y'_k = \sum_{\kappa=1}^m Q_{k\kappa}(z) y'_\kappa \quad (k=1,2,\dots,m)$$

have coefficients $Q_{k\kappa}(z)$ which are rational functions of z with coefficients in K . It would suffice to assume that the coefficients of these rational functions are algebraic numbers, as we could then take for K simply any finite extension of the rational field con-

taining all of them.

We shall be concerned with one fixed solution

$$y_1 = f_1(z), \dots, y_m = f_m(z)$$

of our linear differential equation where $f_1(z), \dots, f_m(z)$ are E-functions. In order to be able to apply the previous results, it will for the present be assumed that $f_1(z), \dots, f_m(z)$ are linearly independent over the field of rational functions with complex coefficients; instead it would suffice to consider independence over the field of rational functions with coefficients in K .

I must first prove two wellknown lemmas on systems of homogeneous linear equations.

(I) Let

$$y_k = a_{k1}x_1 + \dots + a_{kq}x_q \quad (k=1, 2, \dots, p)$$

be p linear forms with rational integral coefficients in q unknowns where $p < q$. Put

$$A = \max_{k, l} |a_{kl}|.$$

There exist rational integers $x_1, \dots, x_q$ not all zero such that

$$y_1 = \dots = y_p = 0, \quad \max(|x_1|, \dots, |x_q|) \leq 1 + (qA)^{\frac{p}{q-p}}.$$

Proof: There is exactly one positive integer H such that

$$(qA)^{\frac{p}{q-p}} - 1 \leq 2H < (qA)^{\frac{p}{q-p}} + 1$$

as the interval has the length 2. Allow $x_1, x_2, \dots, x_q$ to run independently over the $2H+1$ integers $0, \pm 1, \pm 2, \dots, \pm H$. Then

$$-qAH \leq y_k \leq +qAH \quad (k=1, 2, \dots, p)$$

so that y_k has $2qAH+1$ possibilities, hence $(y_1, \dots, y_p)$ has $(2qAH+1)^p$. But there are $(2H+1)^q$ sets $(x_1, \dots, x_q)$, and here

$$(2qAH+1)^p < (qA)^p (2H+1)^p \leq (2H+1)^{q-p} (2H+1)^p = (2H+1)^q.$$

Hence two distinct sets $(x_1' \dots x_q')$ and $(x_1'' \dots x_q'')$ give the same set $(y_1 \dots y_p)$. Put $x_k = x_k' - x_k''$; this set clearly has the asserted properties.

From the last lemma we deduce now a more general one.

(II) Let

$$y_k = a_{k1}x_1 + \dots + a_{kq}x_q \quad (k=1, 2, \dots, p)$$

be p linear forms with integral coefficients in the field K , where

$p < q$. Put

$$A = \max_{k,1} \sqrt[q]{a_{k1}}.$$

There exist integers $x_1, \dots, x_q$ in K not all zero such that $y_1 = \dots = y_p = 0$, $\max (\sqrt[q]{x_1}, \dots, \sqrt[q]{x_q}) \leq c \{1 + (cqA)^{\frac{p}{q-p}}\}$. Here c is a positive constant depending only on K .

Proof: Let $b_1, \dots, b_h$ be an integral basis of K over the rational field; thus any integer a in K can be written as $a = g_1 b_1 + \dots + g_h b_h$ where $g_1, \dots, g_h$ are rational integers. This equation implies

$$a^{(j)} = g_1 b_1^{(j)} + \dots + g_h b_h^{(j)} \quad (j=1, 2, \dots, h)$$

for the conjugates, and here the determinant of the $b_i^{(j)}$ is not zero. Hence, on solving for the g_i ,

$$\max_{i=1, 2, \dots, h} |g_i| \leq \gamma_1 \sqrt[q]{|a|},$$

where $\gamma_1 > 0$ depends only on the chosen basis $b_1, \dots, b_h$ of K .

In the given linear forms we write now each x_k in the form

$$x_k = x_{k1} b_1 + \dots + x_{kh} b_h$$

where the x_{kl} are rational integers; and we also write the new coefficients $a_{kl} b_r$ as linear combinations of $b_1, \dots, b_h$. Then each y_k becomes a linear form in the qh unknowns x_{kl} with coefficients of the form $\sum_{j=1}^h \text{rat. integer } x b_j$, and here the rational integer-coefficients are of absolute value not greater than

$$\gamma_1 \sqrt[q]{a_{kl} b_j} < \gamma_2 A$$

where $\gamma_2 > 0$ depends only on K . Thus

$$y_j = b_1 L_{j1}(x_{kl}) + \dots + b_h L_{jh}(x_{kl}) \quad (j=1, 2, \dots, p)$$

where the $L_{j1}(x_{kl})$ are linear forms with integral coefficients of absolute values less than $\gamma_2 A$. The equations $y_1 = \dots = y_p = 0$ are satisfied exactly when all p h forms $L_{j\tau}(x_{kl})$ vanish. We can now apply lemma I with

$$p, q, A \text{ replaced by } ph, qh, \gamma_2 A$$

and find a solution as required such that

$$\max |x_{kl}| \leq 1 + (\gamma_2 qhA)^{\frac{p}{q-p}}$$

whence the assertion.

Lemma II enables us to construct polynomials $P_1, \dots, P_m$ with not too large integral coefficients in K for which the power series for $P_1 f_1 + \dots + P_m f_m$ begins with a very high power.

Assume the E-functions $f_k(z)$ have the power series

$$f_k(z) = \sum_{\nu=0}^{\infty} c_{k\nu} \frac{z^\nu}{\nu!} \quad (k=1, 2, \dots, m),$$

and let the P_k be polynomials of the form

$$P_k(z) = (2n-1)! \sum_{\nu=0}^{2n-1} g_{k\nu} \frac{z^\nu}{\nu!} \quad (k=1, 2, \dots, m)$$

where the $g_{k\nu}$ are integers in K ; thus P_k is a polynomial at most of degree $2n-1$ with integral coefficients in K ; here n will later be chosen as a very large positive integer. Now

$$P_k(z)f_k(z) = (2n-1)! \sum_{\nu=0}^{\infty} d_{k\nu} \frac{z^\nu}{\nu!} \text{ where } d_{k\nu} = \sum_{\rho=0}^{2n-1} \binom{\nu}{\rho} g_{k\rho} c_{k, \nu-\rho}.$$

The sum

$$R(z) = P_1(z)f_1(z) + \dots + P_m(z)f_m(z) = \sum_{\nu=0}^{\infty} a_\nu \frac{z^\nu}{\nu!}$$

has therefore coefficients

$$a_\nu = (2n-1)! (d_{1\nu} + \dots + d_{m\nu}). \quad (\nu=0, 1, 2, \dots).$$

These coefficients are linear forms in the $m \times 2n$ unknowns $g_{k\nu}$.

Let now ε be an arbitrarily small positive number. We can find a positive integer q_n^* such that all products

$$q_n^* c_{k\nu} \text{ where } k=1, 2, \dots, m \text{ and } \nu=0, 1, \dots, 2mn-n-1$$

are integers in K and that moreover

$$q_n^* = O\left(n^{\frac{2n}{3}}\right);$$

here numbers depending only on m are considered as constants.

We form now the linear equations

$$a_0 = a_1 = \dots = a_{2mn-n-1} = 0;$$

there are $(2m-1)n$ such equations. Each equation we multiply by

$$\frac{q_n^*}{(2n-1)!}$$

It becomes then an equation $L_1(g_{k\rho})=0$ where the coefficient

$$q_n^*(\rho) c_{k,\nu-\rho}$$

of $g_{k\rho}$ lies in K , is an integer, and has all its conjugates of size

$O(n^{\varepsilon n})$
because $\binom{\nu}{\rho} \leq 2^\nu \leq 2^{(2m-1)n} = O(n^{\frac{\varepsilon}{2}n})$ and $|c_{k,\nu-\rho}| = O(n^{\frac{\varepsilon n}{3}})$. We can then apply lemma II to these equations with

$$p=(2m-1)n, \quad q = 2mn, \quad A = O(n^{\varepsilon n}).$$

It follows that, by $\frac{p}{q-p} = 2m-1=O(1)$, there exist integers $g_{k\nu}$ in K satisfying

$$|g_{k\nu}| = O(n^{\varepsilon n}),$$

for which $a_\nu = 0$ if $\nu \leq 2mn-n-1$; here the g 's are not all 0 and hence the same holds for the a 's. Thus we have proved

(III) There exist polynomials $P_1(z), \dots, P_m(z)$ of degree at most $2n-1$ and with integral coefficients in K such that $|a_\nu| = O(n^{(2+\varepsilon)n})$, for which

$$\sum_{k=1}^m P_k(z) f_k(z) = \sum_{\nu=2mn-n}^{\infty} a_\nu \frac{z^\nu}{\nu!} \neq 0$$

and

$$|a_\nu| = \nu^{\varepsilon \nu} O(n^{(2+\varepsilon)n})$$

uniformly in ν .

For $(2n-1)! = O(n^{2n})$, and it is this factor by which everything has been multiplied.

We are assuming from now on that n is sufficiently large; hence the rank 1 has always the value m . Thus if

$$R_k(z) = P_{k1}(z)f_1(z) + \dots + P_{km}(z)f_m(z) \quad (k=1,2,\dots,m),$$

the determinant

$$\Delta(z) = \begin{vmatrix} P_{11}(z), \dots, P_{1m}(z) \\ \vdots \\ P_{m1}(z), \dots, P_{mm}(z) \end{vmatrix}$$

does not vanish identically. On the other hand,

$$\Delta(z)f_k(z) = \sum_{\kappa=1}^m \Delta_{k\kappa}(z)R_\kappa(z) \quad (k=1,2,\dots,m)$$

where the $\Delta_{k\kappa}(z)$ are certain cofactors of $\Delta(z)$, thus are polynomials

Now we found already that, by $V \geq 2mn-n$,
 $\text{ord } R \geq 2mn-n$

and hence

$$\begin{aligned} \text{ord } R_k &\geq 2mn-n-(k-1) \\ &\geq 2mn-n-m+1. \end{aligned}$$

Hence

$$\text{ord } \Delta(z)f_k(z) \geq 2mn-n-m+1.$$

Since $f_k(z) \neq 0$ for all k , the number

$$\min(\text{ord } f_1(z), \dots, \text{ord } f_m(z)) = p \text{ say}$$

is finite, and we have now

$$\text{ord } \Delta(z) \geq 2mn-n-m-p+1.$$

On the other hand, as we also saw, by $U=2n-1$,

$$\deg F_{kk} \leq (2n-1) + (k-1)q$$

and hence

$$\deg \Delta(z) \leq \sum_{k=1}^m \left\{ (2n-1) + (k-1)q \right\} = 2mn-m + \frac{m(m-1)}{2} q.$$

Thus

$$\deg \Delta(z) - \text{ord } \Delta(z) \leq n + \left(p + \frac{m(m-1)}{2} q - 1 \right).$$

This result may be stated in the form

$$\Delta(z) = z^{2mn-n-m-p+1} \Delta_1(z)$$

where now $\Delta_1(z)$ is a polynomial at most of degree

$$n+n_0 \text{ where } n_0 = p + \frac{m(m-1)}{2} q - 1.$$

It is important to note that n_0 is independent of n . We write for shortness

$$N = N(n) = 2mn-n-m-p+1.$$

Then, by what has been proved,

$$z^N \Delta_1(z)f_k(z) = \sum_{\kappa=1}^m \Delta_{k\kappa}(z)R_{\kappa}(z) \quad (k=1, 2, \dots, m).$$

Let now α be an arbitrary number such that

$$\alpha \neq 0, \quad T(\alpha) \neq 0,$$

or more generally,

$$z^N \Delta_1(z) y_k = \sum_{\kappa=1}^m \Delta_{k\kappa}(z) R_{\kappa} \langle y \rangle \quad (k=1, 2, \dots, m)$$

identically for all solutions of our differential equations.

We apply to these identities ν times the operator

$$T(z) \frac{d}{dz}.$$

From $T(z) \frac{d}{dz} R_k \langle y \rangle = R_{k+1} \langle y \rangle$ we then obtain an identity of the form

$$T(z)^\nu \Delta^{(\nu)}(z) y_k + \sum_{N=0}^{\nu-1} \Delta^{(N)}(z) P_{kN} \langle y \rangle = \sum_{\kappa=1}^{m+\nu} \mathcal{J}_{k\kappa}(z) R_{\kappa} \langle y \rangle \quad (k=1, 2, \dots, m).$$

Here the $\mathcal{J}_{k\kappa}(z)$ are certain polynomials, and the $P_{kN} \langle y \rangle$ are certain linear forms in $y_1, \dots, y_m$ with polynomials as coefficients. The proof uses again the differential equations.

Let us now assume that ν is chosen such that

$$\Delta^{(\nu)}(\alpha) \neq 0, \text{ but } \Delta^{(N)}(\alpha) = 0 \text{ if } 0 \leq N < \nu;$$

i.e. α is a zero of the exact order ν of $\Delta(z)$. Clearly

$$0 \leq \nu \leq n+n_0$$

because $\alpha \neq 0$. As also $T(\alpha) \neq 0$, we have then $T(\alpha)^\nu \Delta^{(\nu)}(\alpha) \neq 0$, and hence

$$y_k = \frac{1}{T(\alpha)^\nu \Delta^{(\nu)}(\alpha)} \sum_{\kappa=1}^{m+\nu} \mathcal{J}_{k\kappa}(\alpha) R_{\kappa} \langle y \rangle \Big|_{z=\alpha}.$$

These relations show that we can write

$$y_1, y_2, \dots, y_m$$

as linear combinations with constant coefficients in the $m+\nu$ linear forms

$$R_{\kappa} \langle y \rangle \Big|_{z=\alpha} = P_{\kappa 1}(\alpha) y_1 + \dots + P_{\kappa m}(\alpha) y_m \quad (\kappa=1, 2, \dots, m+\nu).$$

Hence the rank of the latter $m+\nu$ forms over the complex numbers is m , and we can find suffixes

$$k_1, k_2, \dots, k_m \text{ with } 0 \leq k_1 < k_2 < \dots < k_m \leq m+\nu$$

such that

$$\begin{vmatrix} P_{k_1 1}(\alpha), \dots, P_{k_1 m}(\alpha) \\ \vdots \\ P_{k_m 1}(\alpha), \dots, P_{k_m m}(\alpha) \end{vmatrix} \neq 0.$$

The last result holds for all real or complex α with

$$\alpha \neq 0, T(\alpha) \neq 0.$$

Assume from now on that, in addition,

$$\alpha \in K;$$

the numbers $P_{k\kappa}(\alpha)$ ($\kappa, \kappa=1, 2, \dots, m$) become then all elements of K . Our next aim is to study these numbers and the numbers $R_{k\kappa}(\alpha)$. However, it will suffice to consider the numbers

$$P_{k\kappa}(\alpha) \text{ and } R_k(\alpha) \text{ where } 0 \leq k \leq m+\nu \text{ and } 0 \leq \nu \leq n+n_0.$$

The following majorant notations will be used:

$$\sum_0^\infty \alpha_n z^n \prec \prec \sum_0^\infty \beta_n z^n \text{ denotes that } |\alpha_n| \leq \beta_n \text{ for all } n,$$

and

$$\sum_0^\infty \alpha_n z^n \prec \sum_0^\infty \beta_n z^n \text{ denotes that } \alpha_n \in K \text{ and } |\overline{\alpha_n}| \leq \beta_n \text{ for all } n.$$

Denote by c and q two positive integers such that

$$T(z) \prec c(1+z)^q \text{ and } T(z)Q_{k\kappa}(z) \prec c(1+z)^q \text{ for all } k, \kappa;$$

such constants evidently exist. Let us further put

$$\hat{R}(z) = \sum_{\nu=0}^\infty |a_\nu| \frac{z^\nu}{\nu!} \text{ where } R(z) = \sum_{\nu=0}^\infty a_\nu \frac{z^\nu}{\nu!}.$$

We know then already that

$$\hat{R}(z) \prec \prec O(n^{2n}) \sum_{\nu=(2m-1)n}^\infty \nu^{\varepsilon\nu} \frac{z^\nu}{\nu!}.$$

Our aim is to prove that for all $k \geq 0$ and $\kappa=1, 2, \dots, m$,

$$(I) \quad R_{k+1}(z) \prec \prec c^k (1+z)^{kq} \prod_{\nu=0}^{k-1} \left(\nu q + \frac{d}{dz} \right) \hat{R}(z)$$

and

$$(II) \quad P_{k+1,\kappa}(z) \prec c^k (1+z)^{kq+2n-1} \prod_{\nu=0}^{k-1} (\nu q + m + 2n - 1) \cdot O(n^{(2+\varepsilon)n}).$$

From what has been proved, the assertion is certainly true in the lowest case $k=0$. Let it already be proved for $k-1 \geq 0$; we show that it holds then also for k . In the case of

$$R_{k+1}(z) = T(z)R_k^1(z)$$

we find

$$\begin{aligned}
R_{k+1}(z) &< c(1+z)^q \cdot c^{k-1} \left((k-1)q(1+z)^{(k-1)q-1} + (1+z)^{(k-1)q} \frac{d}{dz} \right) \\
&\quad \cdot \prod_{\nu=0}^{k-2} \left(\nu q + \frac{d}{dz} \right) \hat{R}(z) \\
&< c^k (1+z)^{kq} \prod_{\nu=0}^{k-1} \left(\nu q + \frac{d}{dz} \right) \hat{R}(z) \text{ as asserted.}
\end{aligned}$$

Similarly,

$$P_{k+1,\kappa}(z) = T(z) \left(P_{k\kappa}^1(z) + \sum_{\lambda=1}^m P_{k\lambda}(z) Q_{\lambda\kappa}(z) \right)$$

and hence

$$\begin{aligned}
P_{k+1,\kappa}(z) &\prec c(1+z)^q \cdot c^{k-1} \prod_{\nu=0}^{k-2} (\nu q + m + 2n - 1) O(n^{(2n+\varepsilon)n}) \\
&\quad \cdot \left(m + \frac{d}{dz} \right) (1+z)^{(k-1)q+2n-1} \\
&\prec c^k (1+z)^{kq+2n-1} \prod_{\nu=0}^{k-1} (\nu q + m + 2n - 1) O(n^{(2+\varepsilon)n}), \text{ as asserted.}
\end{aligned}$$

We finally use that

$$k \leq m + \nu, \text{ hence } k = N + O(1),$$

and put $z = \alpha$. Then, by $\binom{k}{\rho} \leq 2^k$,

$$\begin{aligned}
\prod_{\nu=0}^{k-1} \left(\nu q + \frac{d}{dz} \right) \hat{R}(z) &< (k-1)! \left(1 + \frac{d}{dz} \right)^k \hat{R}(z) < O(n^{(1+\varepsilon)n}) \left(1 + \frac{d}{dz} \right)^k \hat{R}(z) \\
&< O(n^{(3+\varepsilon)n}) \sum_{\rho=0}^k \sum_{\nu=(2m-1)n}^{\infty} \binom{k}{\rho} \nu^{\varepsilon\nu} \frac{z^{\nu-\rho}}{(\nu-\rho)!} \\
&< O(n^{(3+\varepsilon)n}) \cdot 2^k \sum_{\nu=(2m-1)n-k}^{\infty} (\nu+k)^{\varepsilon(\nu+k)} \cdot \frac{z^{\nu}}{\nu!} \\
&< O(n^{(3+\varepsilon)n}) \cdot 2^k \sum_{\nu=(2m-1)n-k}^{\infty} \left\{ (2\nu)^{2\varepsilon\nu} + (2k)^{2\varepsilon k} \right\} \frac{z^{\nu}}{\nu!}.
\end{aligned}$$

Therefore

$$\prod_{\nu=0}^{k-1} \left(\nu q + \frac{d}{dz} \right) \hat{R}(z) \Big|_{z=\alpha} = O(n^{(3+\varepsilon)n}) \cdot O(n^{\varepsilon n - (2m-2)n}).$$

Therefore, finally,

$$\boxed{|R_k(\alpha)| = O(n^{(3+\varepsilon)n - (2m-2)n})}$$

Since $|\alpha|$ is bounded, the majorant for $P_{k+1,\kappa}(z)$ gives easily the further estimate

$$\boxed{P_{k\kappa}(\alpha) = O(n^{(3+\varepsilon)n})}.$$

These inequalities apply in particular when

$$k = k_1, k_2, \dots, k_m.$$

Let $\Omega_1, \Omega_2, \dots, \Omega_m$ be arbitrary real or complex numbers. These numbers are said to have the exact rank r over the algebraic number field K , if the maximal number of linearly independent homogeneous linear equations

$$\lambda_{k1}\Omega_1 + \dots + \lambda_{km}\Omega_m = 0 \quad (k=1, 2, \dots, R)$$

with coefficients in K is exactly $R=m-r$. Thus r , but not more of the $\Omega_1, \dots, \Omega_m$ are linearly independent over K .

With this definition, we establish now a lower bound for the rank over K of the m numbers

$$f_1(\alpha), f_2(\alpha), \dots, f_m(\alpha),$$

where the conditions on α are as before.

Since $T(\alpha) \neq 0$, at least one of the m numbers $f_1(\alpha), \dots, f_m(\alpha)$ is distinct from zero. Let us assume the rank of these numbers is r , and

$$\lambda_{k1}f_1(\alpha) + \dots + \lambda_{km}f_m(\alpha) = 0 \quad (k=1, 2, \dots, m-r)$$

are the $m-r$ independent relations of K that connect them.

Then, as at least one $f_\kappa(\alpha) \neq 0$,

$$r \geq 1,$$

and we also may assume that all $\lambda_{k\kappa}$ are integers in K .

We apply now the former relations

$$R_k(\alpha) = P_{k1}(\alpha)f_1(\alpha) + P_{k2}(\alpha)f_2(\alpha) + \dots + P_{km}(\alpha)f_m(\alpha) \quad (k=k_1, k_2, \dots, k_m)$$

where, as we know, the determinant

$$\|P_{k\kappa}(\alpha)\| \neq 0.$$

We can then select r of these forms, say those with

$$k = k_1, k_2, \dots, k_r,$$

such that these together with

$$\sum_{\kappa=1}^m \lambda_{k\kappa} f_\kappa(\alpha) \quad (k=1, 2, \dots, m-r)$$

form m linearly independent forms. Put

$$\Lambda = \begin{vmatrix} P_{k_1 1}(\alpha), \dots, P_{k_1 m}(\alpha) \\ \vdots \\ P_{k_r 1}(\alpha), \dots, P_{k_r m}(\alpha) \\ \lambda_{11} & \dots & \lambda_{1m} \\ \vdots & & \vdots \\ \lambda_{m-r1} & \dots & \lambda_{m-rm} \end{vmatrix},$$

so that

$$\Lambda \neq 0.$$

Also denote by $\Lambda_{k\kappa}$ the cofactor of the element in the row and κ the column of Λ , where $k \leq r$; then, in particular,

$$\Lambda f_{\kappa}(\alpha) = \sum_{j=1}^r \Lambda_{j\kappa} R_{k_j}(\alpha) \quad (\kappa=1, 2, \dots, m).$$

Here

$$R_{k_j}(\alpha) = O(n^{(3+\varepsilon)n - (2m-2)n}), \quad |P_{k_j \kappa}(\alpha)| = O(n^{(3+\varepsilon)n}),$$

hence

$$\Lambda_{j\kappa} = O(n^{(3+\varepsilon)n \cdot (r-1)}),$$

hence

$$\Lambda f_{\kappa}(\alpha) = O(n^{(3+\varepsilon)n \cdot r - (2m-2)n}),$$

and

$$|\Lambda| = O(n^{(3+\varepsilon)n \cdot r}),$$

while

$$f_{\kappa}(\alpha) \text{ norm } \Lambda = O(n^{(3+\varepsilon)hrn - (2m-2)n})$$

because

$$f_{\kappa}(\alpha) \text{ norm } \Lambda = \Lambda f_{\kappa}(\alpha) \times \text{product of other conjugates of } \Lambda.$$

Denote by V a positive integer such that

$$V\alpha$$

is an integer in K ; then also

$$g^{2n-1} P_k(\alpha) \text{ and } g^{2n-1+(k-1)q} P_{k\kappa}(\alpha)$$

are integral. It follows therefore from $\Lambda \neq 0$ that

$$g^{O(n)} \text{ norm } \Lambda$$

is an integer and hence

$$|g^{O(n)} \cdot \text{norm } \Lambda| \geq 1.$$

Finally let $n \rightarrow \infty$ and choose κ such that $f_{\kappa}(\alpha) \neq 0$. Then

$$f_{\kappa}(\alpha) = O(n^{(3+\varepsilon)n r h - (2m-2)n}) g^{O(n)}.$$

Here the left-hand side does not depend on n ; hence

$$(3+\varepsilon)rh - (2m-2) \geq 0,$$

and as ε can be arbitrarily small,

$$r \geq \frac{2m-2}{3h}.$$

For $m \geq 4$, the right-hand side is $\geq \frac{\frac{3}{2}m}{3h} = \frac{m}{2h}$. For $m=3$, we have $r \geq \frac{4}{3h}$, since r and h are positive integers, this implies $r \geq \frac{3}{2h} = \frac{m}{2h}$. For $m=1$ or 2 , trivially $r \geq \frac{m}{2h}$.

We have thus proved that always

$$\boxed{r \geq \frac{m}{2h}}.$$

Thus at least $\frac{m}{2h}$ of the numbers

$$f_1(\alpha), \dots, f_m(\alpha)$$

are linearly independent over K .

We can now prove Shidlovski's main result.

We assume that, as before, the E-functions $f_1(z), \dots, f_m(z)$ are a solution of the system of differential equations

$$y'_k = \sum_{\kappa=1}^m Q_{k\kappa}(z) y_\kappa \quad (k=1, 2, \dots, m)$$

where the rational functions $Q_{k\kappa}(z)$ have algebraic coefficients which, without loss of generality, lie in K , just as does the number α for which

$$\alpha \neq 0, \quad T(\alpha) \neq 0.$$

We further assume that there exists no homogeneous polynomial

$$P(z|y_1, y_2, \dots, y_m) \neq 0$$

in $y_1, \dots, y_m$ such that

$$P(z|f_1(z), f_2(z), \dots, f_m(z)) \equiv 0;$$

as function of z , P need not be homogeneous.

Denote by N a large positive integer and put

$$v_{k_1 \dots k_m} = f_1(z)^{k_1} \dots f_m(z)^{k_m} \text{ where } k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m = N;$$

these functions are again E-functions. Their number is

$$\nu = \binom{m-1+N}{m-1} = \frac{(N+m-1)!}{N!(m-1)!}.$$

On differentiating each $v_{k_1, \dots, k_m}$, it becomes obvious that these functions satisfy a homogeneous system of linear differential equations

$$v_{k_1, \dots, k_m}' = \sum_{\kappa_1, \dots, \kappa_m} \overline{\phi}_{k_1, \dots, k_m, \kappa_1, \dots, \kappa_m}(z) v_{\kappa_1, \dots, \kappa_m}$$

where the $\overline{\phi}$ are again rational functions with coefficients in K and with a least common denominator which is power of $T(z)$.

Hence at $z=\alpha$ these functions are regular. It follows then that the rank of the ν numbers $v_{k_1, \dots, k_m}(\alpha)$ over K is not less than

$$\frac{\nu}{2h}.$$

Suppose now our assertion is false, so that

$$P(f_1(\alpha), \dots, f_m(\alpha)) = 0,$$

where $P(y_1, \dots, y_m)$ is a homogeneous polynomial of degree k with coefficients in K . This relation implies all the other ones,

$$f_1(\alpha)^{k_1} \dots f_m(\alpha)^{k_m} P(f_1(\alpha), \dots, f_m(\alpha)) = 0$$

where the k 's are non-negative integers. Let in particular

$$k_1 \geq 0, \dots, k_m \geq 0, \quad k_1 + \dots + k_m = N-k;$$

the last relations are then of dimension N , and their number is

$$\nu^* = \binom{m-1+N-k}{m-1} = \frac{(N+m-k-1)!}{(N-k)!(m-1)!}.$$

It is moreover clear that these new relations, written as linear equations for the $v_{k_1, \dots, k_m}$ are linearly independent. The rank of these $v_{k_1, \dots, k_m}$ is thus not greater than

$$\nu - \nu^*$$

and so

$$\nu - \nu^* \geq \frac{\nu}{2h}.$$

However,

$$\frac{\nu}{\nu^*} = \frac{(N+m-1)!(N-k)!}{(N+m-k-1)!N!} \rightarrow 1 \quad \text{as } N \rightarrow \infty,$$

and hence

$$\nu - \frac{\nu}{2h} < \nu \quad \text{if } N \geq N_0.$$

We thus obtain a contradiction for large N , showing that the assertion must be true.

Let us now replace n by $m+1$ and put

$$f_0(z) \equiv 1.$$

The last theorem leads then immediately to the following further result.

If $f_1(z), \dots, f_m(z)$ satisfy a non-homogeneous system

$$y_k' = Q_{k0}(z) + \sum_{\kappa=1}^m Q_{k\kappa}(z)y_{\kappa} \quad (k=1 \dots m)$$

where the Q 's are rational functions with coefficients in K ; if $\alpha \neq 0$, $T(\alpha) \neq 0$, and $\alpha \in K$; and if $f_1(z), \dots, f_m(z)$ are E-functions which are algebraically independent over the field of rational functions, then $f_1(\alpha), \dots, f_m(\alpha)$ are algebraically independent over the field of rational numbers.

By way of example, if $\omega_1, \dots, \omega_m$ are algebraic numbers that are linearly independent over the rational numbers,

$$e^{\omega_1 z}, \dots, e^{\omega_m z}$$

are E-functions that are algebraically independent over the field of rational functions. Thus $e^{\omega_1}, \dots, e^{\omega_m}$ are algebraically independent over the rational numbers: Lindemann's theorem.

Let $\lambda \neq -1, -2, -3, \dots$ be a rational number such that 2λ is not an odd integer, and let

$$K_{\lambda}(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^n}{n!(\lambda+1)\dots(\lambda+n)}.$$

It can be shown that

$$K_{\lambda}(z) \text{ and } K'_{\lambda}(z)$$

do not satisfy any equation $P(K_{\lambda}, K'_{\lambda}, z) \equiv 0$ where P is a polynomial not identically zero, also $K_{\lambda}, K'_{\lambda}$ satisfy a system of linear differential equations with coefficients which are rational functions. Thus $K_{\lambda}(\alpha), K'_{\lambda}(\alpha)$ are algebraically independent for algebraic $\alpha \neq 0$; for $T(z)$ turns out to be a power of z . This result is due to Siegel; K_{λ} is essentially Bessel's function J_{λ} .